GAUSSIAN PROCESSES
EXERCISE SHEET 10: GAUSSIAN FREE FIELD II

Exercise 1.

Consider the transition probability matrix (P_t) of the continuous-time random walk, defined by

$$(P_t)_{v,w} := \mathbb{P}_v(X_t = w), \qquad v, w \in V \setminus V_\partial.$$

We have

$$P_t = e^{tL},$$

where $L = -\Delta$ and Δ is the discrete Laplacian on the graph.

Thus the Green's function can be written as

$$G = \int_0^\infty e^{tL}\, dt.$$

Step 1: Symmetry. Since L is symmetric, the matrix exponential e^{tL} is symmetric for all t. Hence

$$G = \int_0^\infty e^{tL}\, dt$$

is symmetric.

Step 2: Covariance of the Gaussian Free Field. Recall the identity

$$\frac{d}{dt}\big(e^{tL}L^{-1}\big) = e^{tL}.$$

Integrating from 0 to T yields

$$\int_0^T e^{tL}\, dt = (e^{TL} - I)L^{-1}.$$

Since $e^{TL} \to 0$ as $T \to \infty$ (equivalently, $P_t \to 0$),

$$G = \int_0^\infty e^{tL}\, dt = -L^{-1} = \Delta^{-1}.$$

Therefore G is exactly the covariance matrix of the Gaussian Free Field.

Step 3: Positive definiteness. For any function $\Gamma : V \to \mathbb{R}$ vanishing on V_∂, we have

$$\sum_{v,w \in V \setminus V_\partial} \Gamma(v)\, \Delta_{v,w}\, \Gamma(w) = \frac{1}{2} \sum_{\substack{v,w \in V \\ v \sim w}} (\Gamma(v) - \Gamma(w))^2.$$

The right-hand side is zero if and only if $\Gamma \equiv 0$, so Δ is positive definite, and therefore $G = \Delta^{-1}$ is also positive definite. ☐

Exercise 2.

The discrete Poisson equation on $V \setminus V_\partial$ is

$$\Delta F = f, \qquad F|_{V_\partial} = 0.$$

Since $G = \Delta^{-1}$,

$$F(v) = \sum_{w \in V \setminus V_\partial} G(v, w)\, f(w), \qquad v \in V \setminus V_\partial.$$

Using the probabilistic representation of the Green's function, this becomes

$$F(v) = \mathbb{E}_v \left[\int_0^\tau f(X_t)\, dt \right],$$

which is the unique solution. $\square$

Exercise 3.

(a) Covariance of the massive Gaussian free field.

The massive GFF $\phi = (\phi_x)_{x \in V}$ is defined by the density

$$\mathbb{P}(\phi) \propto \exp\left[-\frac{1}{2} \sum_{x \sim y} (\phi_x - \phi_y)^2 - \frac{m^2}{2} \sum_{x \in V} \phi_x^2 \right] \prod_{x \in V} d\phi_x.$$

Express the quadratic form in matrix notation. Since

$$\sum_{x \sim y} (\phi_x - \phi_y)^2 = \phi^\top \Delta\, \phi,$$

the density becomes

$$\mathbb{P}(\phi) \propto \exp\left[-\frac{1}{2}\, \phi^\top (\Delta + m^2 I) \phi \right] \prod_{x \in V} d\phi_x.$$

Thus ϕ is a centered Gaussian vector with covariance matrix

$$\mathrm{Cov}(\phi) = (\Delta + m^2 I)^{-1},$$

i.e.

$$\mathrm{Cov}(\phi_x, \phi_y) = (\Delta + m^2 I)^{-1}(x, y).$$

(b) Connection to the killed random walk.

Let (X_t) be the continuous-time simple random walk on G, and kill it independently at rate m^2. Let

$$G^m(x, y) = \int_0^\infty \mathbb{P}_x(X_t = y,\ \text{walk not killed before } t)\, dt.$$

Step 1: Green's function via a matrix exponential. Note that if we add a weighted edge with weight m^2 between a new death point v_∂ and all the points in V, we can directly use the idea in Exercise 1 for this problem. (The random walk is killed if and only if it gets to v_∂.) Define

$$L_m := L - m^2 I = -(\Delta + m^2 I).$$

Then the transition matrix of the killed walk is

$$P_t^m = e^{tL_m} = e^{-t(\Delta + m^2 I)},$$

and therefore

$$G^m = \int_0^\infty P_t^m\, dt = \int_0^\infty e^{-t(\Delta + m^2 I)}\, dt.$$

Step 2: Evaluate the integral. Since $\Delta + m^2 I$ is symmetric positive definite,

$$\int_0^\infty e^{-t(\Delta + m^2 I)}\, dt = (\Delta + m^2 I)^{-1}.$$

Thus

$$\mathrm{Cov}(\phi_x, \phi_y) = G^m(x, y).$$

□